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Generalized Galerkin Method for Modal Testing of Structure with Pseudomaterials

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This paper presents a generalized Galerkin method based on the dynamic flexibility matrix expansion technique. The main objective is to develop an analysis to predict the modes and frequencies of a real structure when the design verification methods used to conduct on-the-ground tests can only be performed on a pseudomaterial structure. It is critical to develop analytical methods that correctly predict the dynamic behavior of real structure in order to guarantee a successful design of complex structures. Oftentimes, a full-scale spacecraft modal test cannot be performed on the real model due to hazardous payloads it carries. Stable and relatively inexpensive materials are often used to replace these hazardous materials during the test. The basic idea for analyzing these test modes of pseudomaterial structures is based on eigenvalue reanalysis techniques. This method is relatively easy to understand and can be implemented into the existing method. It is numerically accurate and efficient and can be applied to assist in large, complex space structure designs.

Nomenclature

$F_0(\lambda_0^i)$	=	dynamic flexibility matrix of structure with pseudomaterial
K, M	=	stiffness and mass matrices of structure with real material
K_r, M_r	=	condensed stiffness and mass matrices
K_0, M_0	=	stiffness and mass matrices of structure with pseudomaterial
q^i	=	participation factor column vector, i.e., eigenvector of condensed eigenequation
u^i	=	ith mode (eigenvector) of structure with real material
u_0^i	=	ith mode (eigenvector) of structure with pseudomaterial
$u_1^i, u_2^i, \ldots u_s^i$	=	trial solution vectors of ith mode
λ^i	=	ith eigenvalue of structure with real material
λ_r^i	=	eigenvalues of condensed eigenequation
$\lambda_{r,\text{real}}^i$	=	real eigenvalue identified by condensed (testing) eigenequation
λ_0^i	=	ith eigenvalue of structure with pseudomaterial

I. Introduction

STRUCTURAL dynamic analytical models of spacecrafts require analysis that uses frequency responses as a major design consideration [1–3]. The acceptance of an analytical model is often based on a subjective evaluation of the predicted and measured natural frequencies and mode shapes. The accuracy of the test data and modeling are both important factors in the design and testing processes [2–5].

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Oftentimes, modal testing cannot be performed on a full-scaled real model because its structure and cargo contain toxic, hazardous, or explosive materials. Stable and relatively inexpensive materials are often used to replace these dangerous materials during ground tests. In addition to hazardous material conditions, a full-size spacecraft structure may contain extremely rare or highly valuable materials in their systems. To reduce the cost, these expensive materials used in the test article can be replaced with similar but inexpensive materials. This situation is especially true when engineers deal with ground dynamic tests that do not have the budget for large and complex spacecraft structures.

Pseudostructures can also be used in place of real structures that have extremely complex geometry, shape, and property distribution. It is not easy for engineers to correctly incorporate these shapes and property distributions into their ground test model. To obtain a reasonable representation in pseudomodel, engineers can find ways to relax or modify the real mass and stiffness distribution. A pseudomodel is therefore established which may lead to a very different mass and stiffness matrices. This pseudomodel is not exactly the same as that used in the original model, but will generate similar dynamic characteristics.

A common question is usually raised by the management regarding how to convert the modal test data of a pseudomaterial structure to predict the modes and frequencies of a real structure. A generalized Galerkin method based on the dynamic flexibility matrix expansion technique is developed in this paper to deal with this issue. The fundamental idea for processing these testing modes of a pseudomaterial structure is based on the eigenvalue reanalysis methods [6–11].

In the field of static and eigenvalue reanalyses, an efficient combined approximation (CA) method published by Kirsch is based on the expansion of static flexibility matrix of a structure [6,7]. Using Kirsch's combined method and the Rayleigh quotient technique, Chen and Yang presented a new approach for eigenvalue reanalysis of modified structures [8]. Accurate solutions are obtained by Kirsch and Papalambros's CA method when a basis vector is a linear combination of the previous vectors [10].

The concept for the generalized Galerkin method based on the dynamic flexibility matrix expansion [12–16] technique is quite different from the method presented in the Kirsch's paper, but some of the assumptions, like $\lambda^i \approx \lambda_0^i$, used in the Kirsch's method have been adopted here to find approximated solutions of the system [6,7,10]. The main purpose of this paper is to develop an analytical

method to predict the modes and frequencies of a real structure when all design verification methods used to conduct on-the-ground tests can only be performed on a pseudomaterial structure. The Kirsch's CA method is efficient, and is based on the expansion of the static flexibility matrix approach, whereas ours uses the dynamic flexibility matrix as the basis. Kirsch also makes the displacement assumption that $u^i \approx u_0^i$ in his analysis, whereas the generalized Galerkin method does not make such an assumption; therefore, the generalized Galerkin method arrives at a more precise solution. The basic requirement for the eigenreanalysis method is to possess the feature of fast analysis. For the analysis used here, the primary concern is the accuracy of the method in addition to the fast analysis. This method is relatively easy to understand and can be implemented into existing computer programs.

II. Basis Equation

To clearly define the technical approach forming in the analysis, a pseudomaterial structure is defined as an original system called system A and the corresponding real-material structure is defined as a modified system called system B.

Let the eigenequation of the pseudomaterial system A with n degrees of freedom (DOFs) be

$$K_0 u_0^i = \lambda_0^i M_0 u_0^i, \qquad i = 1, 2, \cdots n \tag{1}$$

where the upperscript i stands for ith eigenpair. The corresponding eigenequation of real-material system B is

$$K u^i = \lambda^i M u^i, \qquad i = 1, 2, \cdots n \tag{2}$$

in which

$$K = K_0 + \Delta K, \qquad M = M_0 + \Delta M \tag{3}$$

Substituting Eq. (3) into Eq. (2) gives

$$(K_0 - \lambda_0^i M_0) u^i = -(\Delta K - \lambda_0^i \Delta M) u^i + (\lambda^i - \lambda_0^i) M u^i \tag{4}$$

On the righthand side of Eq. (4) λ^i is unknown, thereby adopting Kirsch's assumption $\lambda^i \approx \lambda_0^i$ is feasible for solving u^i in terms of Eq. (4) [7,10]. Thus, Eq. (4) is changed to

$$(K_0 - \lambda_0^i M_0) u^i = -(\Delta K - \lambda_0^i \Delta M) u^i \tag{5}$$

Equation (5) is the basis equation used in this paper. Assuming the dynamic flexibility matrix of the pseudomaterial system A is

$$F_0(\lambda_0^i) = (K_0 - \lambda_0^i M_0)^{-1} \tag{6}$$

The matrix inversion shown in Eq. (6) is only used for mathematical description. There is no physical matrix inversion required in the actual computation. The practical computation formula of $F_0(\lambda_0^i)$ is given in the Appendix. It is also defined as

$$X^i = X = -F_0(\lambda_0^i)(\Delta K - \lambda_0^i \Delta M) \tag{7}$$

The elimination of Kirsch's second assumption $u^i \approx u_0^i$ in this derivation increases the accuracy of the identification. Equation (5) can be rewritten as the following iteration format:

$$u_{(p)}^i = X u_{(p-1)}^i, \qquad p = 1, 2, \cdots \tag{8a}$$

$$u_{(0)}^i = u_0^i \tag{8b}$$

Review the iteration process of Eq. (8) as follows:

$$p = 1\text{: } u_{(1)}^i = X u_{(0)}^i = X u_0^i \tag{9a}$$

$$p = 2\text{: } u_{(2)}^i = X u_{(1)}^i = X^2 u_0^i \tag{9b}$$

$$p = 3\text{: } u_{(3)}^i = X u_{(2)}^i = X^3 u_0^i \tag{9c}$$

From Eq. (9), another iteration formulation of Eq. (8) can be induced as

$$u_{(p)}^i = X^p u_0^i, \qquad p = 1, 2, \cdots \tag{10}$$

Here the upperscript p stands for the pth power of X.

III. Generalized Galerkin Method

A. Assembly of Trial Solution Vectors and Galerkin Equation

From the analytical derivation, Eq. (5) shown in this paper is the fundamental working function. The homogeneous solution of Eq. (5) is u_0^i. In addition, the coefficient matrix is singular. This characteristics gives an infinite number of particular solutions which lead to an infinite number of general solutions. To find a special unique particular solution for engineers to use, one can use the perturbation method, shown in the Appendix, to eliminate the singularity from Eq. (5). Then, a special particular solution can be obtained as

$$u^i = X u^i \tag{11}$$

To find the solution for the above-mentioned Eq. (11), one has to use Eq. (8), or Eq. (10), as an iteration formula. From Eq. (10), it is well known that the present iteration process is divergent when $\|X\| > 1$. This process may also converge to a nonreasonable numerical result when $\|X\| < 1$. To find a unique solution of the system, we choose to use the Galerkin method. We also use the homogeneous solution u_0^i of Eq. (5) and the first $(s-1)$ number times of the iteration results $(Xu_0^i, X^2u_0^i, X^3u_0^i \ldots X^{s-1}u_0^i)$ of Eq. (10) as trial solution vectors $(u_1^i, u_2^i, \ldots u_s^i)$. The first several iteration results of Eq. (10) can be considered as initial approximation values of the particular solutions of Eq. (5).

According to the principle of Galerkin method, u^i can be expressed as a linear combination of these trial solution vectors:

$$\begin{aligned} u^i &= q_1^i u_1^i + q_2^i u_2^i + \cdots + q_s^i u_s^i = q_1^i(u_0^i) + q_2^i(Xu_0^i) \\ &\quad + q_3^i(X^2u_0^i) + \cdots + q_s^i(X^{s-1}u_0^i) = V_B^i q^i \end{aligned} \tag{12}$$

in which

$$V_B^i = [u_1^i, u_2^i, \cdots u_s^i] = [u_0^i, Xu_0^i, X^2u_0^i, \cdots X^{s-1}u_0^i] \tag{13}$$

$$q^i = \{q_1^i, q_2^i, \cdots q_s^i\} \tag{14}$$

Here q^i is defined as a generalized coordinate vector (participation factor column vector). Substituting Eq. (12) into Eq. (2) and requiring $V_B^{iT}(KV_B^i q^i - \lambda^i M V_B^i q^i)$ equal to zero result in a Galerkin equation. This is a reduced eigenequation of system B:

$$K_r^i q^i = \lambda_r^i M_r^i q^i \tag{15}$$

where

$$K_r^i = V_B^{iT} K V_B^i, \qquad M_r^i = V_B^{iT} M V_B^i \tag{16}$$

To improve the precision of numerical results, sometimes, it may require to use the Gram-Schmidt orthogonalization procedure to make trial solution vectors $(u_1^i, u_2^i, \ldots u_s^i)$ orthogonal.

B. Reason Analysis and Characteristics of Trial Solution Vectors

The reason for using these trial solution vectors is presented here in terms of concept of the CA method as described in [6–10]. The following equality can be written from a linearly combined formula of Eq. (12):

$$u^i = u_0^i + Xu_0^i + X^2u_0^i + \cdots + X^{s-1}u_0^i \tag{17a}$$

$$= [I + X + X^2 + \cdots + X^{s-1}]u_0^i = [I - X]^{-1}u_0^i \tag{17b}$$

i.e.

$$[I - X]u^i = u_0^i \tag{18}$$

Substituting Eq. (7) into Eq. (18) yields

$$[I + F_0(\lambda_0^i)(\Delta K - \lambda_0^i \Delta M)]u^i = u_0^i \tag{19}$$

Embedding Eq. (6) into Eq. (19) gives

$$(K - \lambda_0^i M)u^i = (K_0 - \lambda_0^i M_0)u_0^i = 0 \tag{20}$$

In addition, the original Eq. (4) corresponding to Eq. (20) can be written as

$$(K - \lambda_0^i M)u^i = (\lambda^i - \lambda_0^i)Mu^i \tag{21}$$

If taking the $\lambda^i \approx \lambda_0^i$ assumption, Eq. (21) becomes

$$(K - \lambda_0^i M)u^i = 0 \tag{22}$$

A comparison between Eq. (20) and (22) shows that the trial solution vectors taken in Eq. (12) are reasonable.

From the physical point of view, u^i is the real-material structural dynamic modal testing mode shape. u_0^i is the homogeneous solution and is the basic mode shape of u^i. Those particular solutions $Xu_0^i, X^2u_0^i, X^3u_0^i \ldots X^{s-1}u_0^i$ are the real-material structural mode shape corrections which are corresponding to the pseudomaterial structure.

C. Solving the Special Eigenequation

The Galerkin equation shown in Eq. (15) is a very special eigenequation. All its s eigenvalues λ_r^i are theoretical values of λ^i. In the practical application, only one λ_r^i value in these s eigenvalues, as well as its corresponding q^i, are useful and usable. To select this useful/usable eigenpair (λ_r^i, q^i), one can develop a criteria for selecting proper eigenpairs. Chen and Yang proposed a criteria [8] that the smallest value in all s number of λ_r^i is the ith eigenvalue of the system with real material. This criteria is not true in general. For this reason, we develop a selection criteria for choosing proper λ_r^i. Based on Eq. (12), using s generalized coordinate vectors q^i computed by Eq. (15), one finds s column vectors $u_{(j)}^i (j = 1, 2, \ldots s)$ first. Then, calculating the following s cosine values

$$\cos\beta_{(j)}^i = \frac{(u_{(j)}^{iT} u_0^i)}{(|u_{(j)}^i||u_0^i|)}, \qquad j = 1, 2, \cdots s \tag{23}$$

Here $\beta_{(j)}^i$ stands for the angle between $u_{(j)}^i$ and u_0^i. An $u_{(j)}^i$ that causes $\cos\beta_{(j)}^i$ $(j = 1, 2, \ldots s)$ to approach 1 is just identified as an ith testing mode u^i of the real-material structure. After determining u^i, the ith testing frequency (or eigenvalue) λ_t^i of the real-material structure corresponding to ui is found by the following Rayleigh quotient:

$$\lambda_t^i = \frac{u^{iT} K u^i}{u^{iT} M u^i} \tag{24}$$

It is important to point out that because a part of CA method concept is used in the reason analysis of trial solution vectors and the derived Galerkin equation is also a very special eigenequation, the aforementioned Galerkin method is called *generalized Galerkin method*.

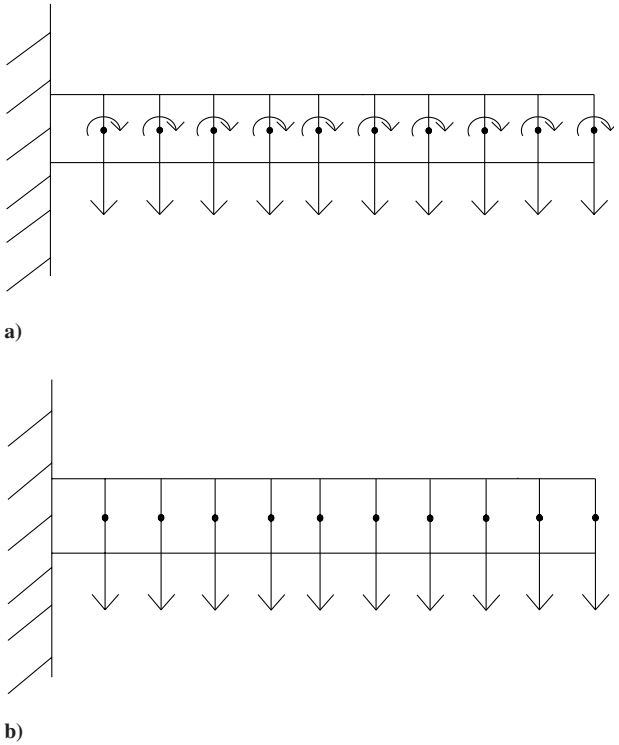

Fig. 1 Simple and condensed cantilever beam models: a) simple, with 20 DOF and b) condensed, with 10 DOF.

IV. Numerical Example

To verify the generalized Galerkin method, a numerical simulation example is presented here. This example is a simple cantilever beam which has pure-flexure in the vertical plane and possesses 10 finite elements. It has 10 transverse displacements and 10 flexure motions as shown in Fig. 1a. This example has 20 DOFs. To reduce DOFs in the calculation, use condensation technique to condense 10 flexure DOFs and let rotational inertia at flexure DOF be zero for simplification. The final condensed cantilever beam finite element model has 10 DOFs as shown in Fig. 1b.

The elastic module of the real-material (modified) structure and the pseudomaterial (original) structure are 70 and 210 Gpa, respectively. Their masses are embodied by the lump-masses. The unit of the mass is kilogram (Kg). Size of lump-masses at every node emerges at diagonal elements of the mass matrices as shown in Eqs. (25b) and (26b). Stiffness and mass matrices (K and M) of real-material structure are listed, respectively, in Eqs. (25a) and (25b). Correspondingly, K_0 and M_0 of pseudomaterial structure are given in Eqs. (26a) and (26b). The eigenpairs of real-material and pseudomaterial structures obtained from Eqs. (1) and (2) are listed in Tables 1 and 2. The values shown in the brackets of Tables 1 and 2 are eigenpair data of pseudomaterial structure:

$$K = \begin{bmatrix} 4.8871 & -2.9605 & 1.0137 & -0.1722 & 0.0292 & -0.0050 & 0.0008 & -0.0001 & 0 & 0 \\ & 3.9302 & -2.7980 & 0.9861 & -0.1675 & 0.0284 & -0.0048 & 0.0008 & -0.0001 & 0 \\ & & 3.9026 & -2.7933 & 0.9853 & -0.1674 & 0.0284 & -0.0048 & 0.0008 & -0.0001 \\ & & & 3.9018 & -2.7932 & 0.9853 & -0.1673 & 0.0284 & -0.0047 & 0.0006 \\ & & & & 3.9018 & -2.7931 & 0.9853 & -0.1672 & 0.0276 & -0.0034 \\ & & & & & 3.9017 & -2.7930 & 0.9845 & -0.1625 & 0.0202 \\ & & & & & & 3.9009 & -2.7883 & 0.9569 & -0.1188 \\ & \text{symmetry} & & & & & & 3.8733 & -2.6258 & 0.6992 \\ & & & & & & & & 2.9164 & -1.1086 \\ & & & & & & & & & 0.5109 \end{bmatrix} \times 10^7 \tag{25a}$$

$$M = \begin{bmatrix} 8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & 8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & 8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & & 16 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & & & 15 & 0 & 0 & 0 & 0 & 0 \\ & & & & & 8 & 0 & 0 & 0 & 0 \\ & & & & & & 16 & 0 & 0 & 0 \\ & & \text{symmetry} & & & & & 8 & 0 & 0 \\ & & & & & & & & 8 & 0 \\ & & & & & & & & & 8 \end{bmatrix} \tag{25b}$$

$\lambda_r^i = 4.0822\text{E}+2$ for the first mode, $\lambda_r^i = 1.3622\text{E}+4$ for the second mode, and $\lambda_r^i = 1.1159\text{E}+5$ for the third mode. These significant real λ_r^i values are named as $\lambda_{r,\text{real}}^i$. Modified values λ_t^i of these significant real $\lambda_{r,\text{real}}^i$ are determined by Eq. (24). Also, its corresponding eigenvalue calculated by Eq. (2) is defined as λ^i, and is considered the exact value. Thus, three types of eigenvalues are listed in Table 5. The percentage error $\lambda_t^i\%$ is defined as

$$\lambda_t\% = \frac{\lambda^i - \lambda_t^i}{\lambda^i}\% \tag{27}$$

$$K_0 = \begin{bmatrix} 1.4661 & -0.8882 & 0.3041 & -0.0517 & 0.0088 & -0.0015 & 0.0003 & 0 & 0 & 0 \\ & 1.1792 & -0.8394 & 0.2958 & -0.0502 & 0.0085 & -0.0014 & 0.0002 & 0 & 0 \\ & & 1.1708 & -0.8380 & 0.2956 & -0.0502 & 0.0085 & -0.0014 & 0.0002 & 0 \\ & & & 1.1705 & -0.8379 & 0.2956 & -0.0502 & 0.0085 & -0.0014 & 0.0002 \\ & & & & 1.1705 & -0.8379 & 0.2956 & -0.0502 & 0.0083 & -0.0010 \\ & & & & & 1.1705 & -0.8379 & 0.2954 & -0.0844 & 0.0061 \\ & & & & & & 1.1703 & -0.8365 & 0.2871 & -0.0356 \\ & & \text{symmetry} & & & & & 1.1620 & -0.7877 & 0.2098 \\ & & & & & & & & 0.8749 & -0.3326 \\ & & & & & & & & & 0.1533 \end{bmatrix} \times 10^8 \tag{26a}$$

$$M_0 = \begin{bmatrix} 10 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & 10 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & 10 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & & 10 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & & & 10 & 0 & 0 & 0 & 0 & 0 \\ & & & & & 10 & 0 & 0 & 0 & 0 \\ & & & & & & 10 & 0 & 0 & 0 \\ & & \text{symmetry} & & & & & 10 & 0 & 0 \\ & & & & & & & & 10 & 0 \\ & & & & & & & & & 10 \end{bmatrix} \tag{26b}$$

Based on the generalized Galerkin method, the modal test data of pseudomaterial structure (such as ith eigenvector u_0^i) is used to find its corresponding ith eigenvector of the real-material structure u^i. We also use four trial solution vectors u_0^i, Xu_0^i, $X^2u_0^i$, $X^3u_0^i$ in the calculation. The $F_0(\lambda_0^i)$ contained in the X is taken up to the first four terms of Eq. (A2) and including the A_2 term. Next, the linear combination formula of u^i is established by using Eq. (12). Further, four generalized eigenpairs (λ_r^i, q^i) are found by using Eq. (15). Only the first three test modes of the real-material structure are identified. Thus, λ_r^i values with $i = 1, 2, 3$ are listed in Table 3. The comparison between data listed in both Tables 1 and 3 indicates another interesting finding. Previously published paper [8] shows that a criterion based on the engineers' experiences (namely the smallest eigenvalue $\lambda_{r,\min}^i$ of Eq. (15) is the only λ^i identified) does not hold true. The criterion embodied by Eq. (23) is used to select real λ^i. For this reason, $\cos\beta_{(j)}^i$ values computed by Eq. (23) are given in Table 4.

Table 4 clearly indicates the highest $\cos\beta_{(j)}^i$ values close to 1 used in the judgment criterion for each mode. From these highest $\cos\beta_{(j)}^i$ values, the corresponding real λ^i values, shown in Table 3, are

Testing modes of the real-material structure, which are identified by the generalized Galerkin method, are shown in Table 6, in which the values in the brackets represent the exact values calculated by Eq. (2). The percentage error $u_t^i\%$ is defined as

$$u_t^i\% = \frac{\|u^i - u_t^i\|}{\|u^i\|}\% \tag{28}$$

where u_t^i stands for real-material structural testing mode identified by the generalized Galerkin method.

Table 2 Eigenvectors of real-material (pseudomaterial) structure

Eigenvectors $u^i(u_0^i)$					
$i=1$	$i=2$	$i=3$	$i=4$	$i=5$	$i=6$
0.0170	−0.0895	0.2441	−0.8659	0.9337	−1.4423
(0.0165)	(−0.1128)	(0.3403)	(−0.7173)	(1.2690)	(−2.0172)
0.0643	−0.2885	0.6593	−1.7904	1.3208	−1.2321
(0.0627)	(−0.3639)	(0.8968)	(−1.4068)	(1.5847)	(−1.0826)
0.1370	−0.5077	0.8849	−1.3846	0.0879	1.0176
(0.1340)	(−0.6426)	(1.1617)	(−0.9229)	(−0.3114)	(1.9076)
0.2304	−0.6663	0.7122	0.1153	−0.9461	1.0005
(0.2261)	(−0.8525)	(0.8946)	(0.4059)	(−1.6400)	(0.6810)
0.3400	−0.7004	0.1535	1.2394	0.0251	−1.4649
(0.3348)	(−0.9231)	(0.1954)	(1.2949)	(−0.3647)	(−2.0415)
0.4613	−0.5820	−0.4882	0.8076	1.1768	−0.3766
(0.4560)	(−0.8167)	(−0.5744)	(0.8703)	(1.4934)	(−0.3202)
0.5908	−0.3189	−0.8301	−0.4853	0.5462	1.3342
(0.5860)	(−0.5322)	(−0.9956)	(−0.4194)	(1.0121)	(2.1035)
0.7251	0.0613	−0.6183	−1.1698	−1.1789	−0.6671
(0.7216)	(−0.1000)	(−0.8139)	(−1.2337)	(−1.0234)	(−0.0258)
0.8621	0.5151	0.0703	−0.5505	−1.1902	−1.7192
(0.8602)	(0.4289)	(−0.0611)	(−0.6538)	(−1.3033)	(−1.9680)
1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
(1.0000)	(1.0000)	(1.0000)	(1.0000)	(1.0000)	(1.0000)

Table 1 Eigenvalues of real-material (pseudomaterial) structure

Values $\lambda^i(\lambda_0^i)$					
$i=1$	$i=2$	$i=3$	$i=4$	$i=5$	$i=6$
4.0822E + 2	1.3622E + 4	1.1159E + 5	4.6645E + 5	1.2054E + 6	2.2373E + 6
(1.1418E + 3)	(4.4814E + 4)	(3.4801E + 5)	(1.3089E + 6)	(3.4514E + 6)	(7.2768E + 6)

Table 3 Eigenvalues λ_r^i of identified eigenequation (15)

λ_r^i		
$i=1$	$i=2$	$i=3$
4.0822E + 2	4.1074E + 2	1.3213E + 4
1.3624E + 4	1.3622E + 4	3.5053E + 4
1.1173E + 5	1.1206E + 5	1.1159E + 5
2.4835E + 6	6.2454E + 5	5.8587E + 5

Table 4 $\cos\beta_{(j)}^i$ values used in the judgment criterion

	$\cos\beta_{(j)}^i$		
Order	$i=1$	$i=2$	$i=3$
$j=1$	0.9999881	0.0049104	0.0103929
$j=2$	0.1878153	0.9821481	0.0126878
$j=3$	0.0802486	0.0837946	0.9922468
$j=4$	0.1343345	0.1708508	0.2253234

Table 5 Three types of eigenvalues for real-material structure

	Three types of eigenvalues and % errors			
	λ^i	$\lambda_{r,\text{real}}^i$	λ_t^i	$\lambda_t^i\%$
$i=1$	4.0822E + 2	4.0822E + 2	4.0822E + 2	−0.260E − 7
$i=2$	1.3622E + 4	1.3622E + 4	1.3622E + 4	−0.364E − 3
$i=3$	1.1159E + 5	1.1159E + 5	1.1159E + 5	−0.494E − 2

From the data shown in Tables 5 and 6, it is found that the precision of the generalized Galerkin method developed in this paper is satisfactory for engineering application.

V. Discussions

1) The generalized Galerkin method based on the dynamic flexibility matrix expansion approach is different from the Kirsch method [6–10]. This Galerkin method adopts a part of the basis vectors combination method concept used in the Kirsch method. Therefore, it is called a generalized Galerkin method. Like the Kirsch method, the generalized Galerkin method can also be used in the reanalysis of eigenproblems.

Table 6 Identified testing modes and % errors for real-material structure

	$u_t^1(u^1)$	$u_t^2(u^2)$	$u_t^3(u^3)$
	0.0168747	−0.0894943	0.2445958
	(0.0170)	(−0.0895)	(0.2441)
	0.0642526	−0.2885692	0.6605087
	(0.0643)	(−0.2885)	(0.6593)
	0.1370013	−0.5078038	0.8868235
	(0.1370)	(−0.5077)	(0.8849)
	0.2304476	−0.6663264	0.7144475
	(0.2304)	(−0.6663)	(0.7122)
	0.3399661	−0.7004068	0.1533451
	(0.3400)	(−0.7004)	(0.1535)
	0.4613184	−0.5820943	−0.4874771
	(0.4613)	(−0.5820)	(−0.4882)
	0.5907844	−0.3190688	−0.8259988
	(0.5908)	(−0.3189)	(−0.8301)
	0.7251453	0.0614417	−0.6100489
	(0.7251)	(0.0613)	(−0.6183)
	0.8620885	0.5153523	0.0768401
	(0.8621)	(0.5151)	(0.0703)
	1.0000000	1.0000000	1.0000000
	(1.0000)	(1.0000)	(1.0000)
$u_t^i\%$	0.104E − 10	0.243E − 5	0.169E − 2

2) The accuracy of both the generalized Galerkin method and the Kirsch basis vector combination method is discussed. Using Kirsch's assumption $\lambda^i \approx \lambda_0^i$ and $u^i \approx u_0^i$ results in the following characteristics of the eigenfunction:

$$Ku^i = \lambda_0^i M u_0^i \tag{29}$$

Similarly, this paper uses generalized Galerkin method Eq. (5) to find the following approximation characteristics of the eigenfunction:

$$Ku^i = \lambda_0^i M u^i \tag{30}$$

Comparing Eq. (29) and (30), one can conclude that the generalized Galerkin method will yield better results as compared with the Kirsch method.

3) In the eigenvalue reanalysis design, both ΔK and ΔM matrices are normally known values in the study. This means that both K and M matrices are also known. However, this situation is quite different when a reanalysis method is employed in the processing of pseudomaterial structural testing modes. Under this situation, both ΔK and ΔM matrices are unknown. There are two ways to deal with this situation: a) use a combination of test and analysis data to arrive at an appropriated value for the ΔK and ΔM matrices and b) deal with more complex structures, one can use the modal testing data obtained from the pseudosystem, using a design parameter-type method [12] combined with a model updating technique to modify both K_0 and M_0 matrices. It is then possible to use the real-material design parameters (such as E, ρ, ... etc.) to replace the corresponding pseudomaterial design parameters (E_0, ρ_0, ... etc.). One can immediately acquire better K and M matrices information, thereby obtaining better ΔK and ΔM matrices. This is because the design parameter-type model updating method does not, in general, modify the parameters related to the materials. The design parameter (data) of materials used in both K_0 and M_0 matrices are correct and cannot be modified.

4) From Eq. (A3), one knows that the matrices $A_p (p \geq 0)$ needs to be calculated only once in order to identify the k modes $u^i (i = 1, 2, \cdots k)$ of a real-material system. When there is a rigid-body motion, K_0 is a singular matrix. It is well known that using a unitized shift-frequency technique [12] can eliminate the existing singularity. The banded-state matrix merit of K_0 has also been kept. However, when K_0 is a singular matrix, the calculation of real-material system matrix, shown in Kirsch method, will be inconvenient.

5) Because of the fact that the participation factors $q_1^i, q_2^i, \cdots q_s^i$ have the ability to fine tune the final results; the generalized Galerkin method still maintains better precision for its results even when λ^i differs significantly from λ_0^i.

VI. Conclusions

1) A generalized Galerkin method based on the dynamic flexibility matrix method has been developed. This method provides accurate results and no matrix inversion is required during the computation.

2) The generalized Galerkin method uses pseudostructure dynamic flexibility matrices to represent $F_0(\lambda_0^i)$ value in the analysis, in addition to pseudostructure dynamic modal testing parameters u_0^i and λ_0^i.

3) The generalized Galerkin method can also be used for all types of eigenvalue reanalysis task. From previous experiences, numerical solution for choosing the perturbation constant η value between 0.001 to 0.0001 is adequate for engineers to find good solutions.

4) This method addresses how to choose an initial trial vector to find the final solution. This paper gives a very detail mathematical and physical explanation to use the initial trial solution vector.

5) This method is relatively easy to understand and can be implemented into the existing method. This method is numerically efficient and accurate and can be applied to future large complex spacecraft structure design.

Appendix: Singular Dynamic Flexibility Matrix

To derive a generalized governing equation to cover a broader range of conditions encountered in the engineering application, let us assume λ_0^i be a m-multiple root (eigenvalue) while discussing the $F_0(\lambda_0^i)$ formula used in this paper. When λ_0^i is a single root equal to$\bar{\lambda}_0^i$, obtaining $F_0(\bar{\lambda}_0^i)$ value is only a special case of the $F_0(\lambda_0^i)$ formula with a m-multiple root.

Let us define a system with a lower-order eigenpair assembly (constructed by k lower-order eigenpairs) of a pseudomaterial system A and it is defined as U_{0k} and Λ_{0k}. The matrix Λ_{0k} includes m-multiple eigenvalues λ_0^i. All eigenvectors corresponding to λ_0^i are $Z_0^i \in R^{n,m}$. Thus, there are

$$U_{0k} = [\bar{U}_{0k}, Z_0^i], \qquad \Lambda_{0k} = \text{diag}[\bar{\Lambda}_{0k}, \lambda_0^i I] \tag{A1}$$

in which $\bar{U}_{0k}$ is the remainder part of U_{0k} except for Z_0^i.

The dynamic flexibility matrix $F_0(\lambda_0^i)$ used in Eq. (6) of a pseudomaterial system A is a singular matrix. To compute the result, taking its perturbed dynamic flexibility matrix $F_0(\tilde{\lambda}_0^i)$ approach will give a better approximation result. The perturbed dynamic flexibility matrix of system A is [14–16]

$$F_0(\tilde{\lambda}_0^i) = \bar{U}_{0k}(\bar{\Lambda}_{0k} - \tilde{\lambda}_0^i I)^{-1}\bar{U}_{0k}^T + A_0 + \tilde{\lambda}_0^i A_1 + (\tilde{\lambda}_0^i)^2 A_2 + \cdots \tag{A2}$$

Here the perturbed eigenvalue is written as $\tilde{\lambda}_0^i = \lambda_0^i + \varepsilon$. ε is a small constant that is chosen from previous experience by selecting the criterion $\eta = \varepsilon/\lambda_0^i = 0.001 - 0.0001$. This selection results in $\tilde{\lambda}_0^i = (1+\eta)\lambda_0^i$. In addition, matrices $A_p (p = 0, 1, \cdots)$ are found from the following equations:

$$K_0 A_0 = I - M_0 U_{0k} U_{0k}^T \tag{A3a}$$

$$K_0 A_p = M_0 A_{p-1}, \qquad p = 1, 2, \cdots \tag{A3b}$$

Equation (A3) needs merely to be solved once, no matter what λ_0^i value is given. Because the λ_0^i term does not appear in the Eq. (A3), engineers only need to decouple the system coefficient matrices once during the analysis. This is a very efficient method for calculating $F_0(\tilde{\lambda}_0^i)$ with different λ_0^i.

When testing a free-free structure with a rigid body motion, K_0 is a singular matrix. Equation (A3) cannot be solved. To eliminate the present singularity and still retain the banded-state merit of K_0 matrix, a simple uniform shift-frequency technique [13] can be used. Once $\tilde{\lambda}_0^i \approx \lambda_{k+1}$ and $\tilde{\lambda}_0^i > \lambda_{k+1}$ (λ_{k+1} indicates the smallest eigenvalue in the higher-order eigenvalue assembly Λ_{0k} of system A) conditions occur, the power-series shown in Eq. (A2) diverges [14,15]. Under these situations, the best approach is to use a hybrid shift-frequency technique [15], since the hybrid shift-frequency technique can deal with various nonconvergent conditions. These shift-frequency techniques are not used in this paper, because it is sufficient to use both Eqs. (A2) and (A3) to compute $F_0(\tilde{\lambda}_0^i)$ for the numerical example.

References

[1] Flannely, W. G., Fabunmi, J. A., and Nagy, E. J., "Analytical Testing," NASA CR-3429, May 1981.

[2] Berman, A., and Flannely, W. G., "Theory of Incomplete Models of Dynamic Structures," *AIAA Journal*, Vol. 9, No. 8, 1971, pp. 1481–1487. doi:10.2514/3.49950

[3] Berman, A., and Wei, F. S., "Automated Dynamic Analytical Model Improvement," NASA CR-3452, July 1981.

[4] Berman, A., Wei, F. S., and Rao, K. W., "Improvement of Analytical Models Using Modal Test Data," *AIAA/ASME/ASCE/AHS 21st Structural Dynamics and Materials Conference*, Seattle, WA, May 1980, pp. 809–814.

[5] Zhang, O., Zerva, A., and Zhang, D. W., "Stiffness Matrix Adjustment Using Incomplete Measured Modes," *AIAA Journal*, Vol. 35, No. 5, 1997, pp. 917–919. doi:10.2514/2.7469

[6] Kirsch, U., "Efficient, Accurate Reanalysis for Structural Optimization," *AIAA Journal*, Vol. 37, No. 12, 1999, pp. 1663–1669. doi:10.2514/2.649

[7] Kirsch, U., "Approximate Vibration Reanalysis of Structures," *AIAA Journal*, Vol. 41, No. 3, 2003, pp. 504–511. doi:10.2514/2.1973

[8] Chen, S. H., and Yang, X. W., "Extended Kirsch Combined Method for Eigenvalue Reanalysis," *AIAA Journal*, Vol. 38, No. 5, 2000, pp. 927–930. doi:10.2514/2.1049

[9] Wu, X. M., Chen, S. H., and Yang, Z. J., "Static Displacement Reanalysis of Modified Structures Using the Epsilon Algorithm," *AIAA Journal*, Vol. 45, No. 8, Aug. 2007, pp. 2083–2086. doi:10.2514/1.19767

[10] Kirsch, U., and Papalambros, P. Y., "Exact and Accurate Solutions in the Approximate Reanalysis of Structures," *AIAA Journal*, Vol. 39, No. 11, 2001, pp. 2198–2205. doi:10.2514/2.1219

[11] Liu, J. K., "Universal Perturbation Technique for Reanalysis of Non-Self-Adjoint Systems," *AIAA Journal*, Vol. 38, No. 6, 2000, pp. 1035–1039. doi:10.2514/2.1064

[12] Zhang, D. W., and Wei, F. S., *Model Updating and Damage Detection*, Science Press, Beijing, China, 1999.

[13] Hu, H. C., *Vibration Theory of Multi-Degree of Freedom System*, Science Press, Bejing, 1987, pp. 69–75.

[14] Zhang, D. W., and Wei, F. S., "Efficient Computation of Many Eigenvector Derivatives Using Dynamic Flexibility Method," *AIAA Journal*, Vol. 35, No. 4, 1997, pp. 712–718. doi:10.2514/2.162

[15] Zhang, D. W., and Wei, F. S., "Dynamic Flexibility Method with Hybrid Shifting Frequency for Eigenvector Derivatives," *AIAA Journal*, Vol. 40, No. 10, 2002, pp. 2047–2052. doi:10.2514/2.1537

[16] Zhang, D. W., and Wei, F. S., "Direct Perturbation Methods for Calculation of Eigenvector Derivatives with Repeated Eigenvalues," *Acta Mechanica Solida Sinica*, Vol. 14, No. 4, 1993, pp. 337–341.

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